arXiv:1005.0459v2 [hep-th] 15 Jul 2010

Bound of Noncommutativity Parameter Based on Black Hole Entropy

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ABSTRACT

We study the bound of the noncommutativity parameter in the noncommutative Schwarzschild black hole which is a solution of the noncommutative $ISO(3,1)$ Poincaré gauge group. The statistical entropy satisfying the area law in the brick wall method yields a cutoff relation which depends on the noncommutativity parameter. Requiring both the cutoff parameter and the noncommutativity parameter to be real, the noncommutativity parameter can be shown to be bounded as $\Theta > 8.4 \times 10^{-2} l_p$.

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Since short distance behaviors have been extensively studied, it has been claimed that the description of spacetime as a discontinuous manifold may be plausible. In this respect, it seems to be natural to consider spacetimes related with noncommutativity in which the coordinates become noncommutative. Its notion has been popular in the context of the string theory and intrinsically connected with gravity [1, 2, 3]. Explicitly, canonical commutation relations for noncommutative spacetime are assumed to be

$$[\hat{x}^\alpha, \hat{x}^\beta] = i\Theta^{\alpha\beta}, \tag{1}$$

where $\Theta^{\alpha\beta}$ are anti-symmetric tensors. It has been well known that a theory on noncommutative spacetime along with the commutation relation (1) is equivalent to another theory on commutative spacetime in which a product of any two functions on the original noncommutative spacetime is replaced with the Moyal star($\star$)-product [4, 5]:

$$(f \star g)(x) \equiv \exp\left[\frac{i}{2}\Theta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial y^\beta}\right] f(x)g(y)\bigg|_{x=y}. \tag{2}$$

Such a deformation in a gravity side can be constructed based on gauging the noncommutative $SO(4,1)$ de Sitter group and the Seiberg-Witten map with subsequent contraction to $ISO(3,1)$ Poincaré gauge group [6, 7, 8].

Various spherically symmetric black hole solutions and cosmological solutions in the commutative spacetime are promoted to the noncommutative ones through the Seiberg-Witten map [9, 10, 11, 12]. In particular, the metric of the noncommutative Schwarzschild black hole up to the second order of the noncommutativity parameter is given by [9]

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2, \tag{3}$$

where

$$\begin{aligned}
g_{tt} &= -\left[\left(1 - \frac{2GM}{r}\right) + \frac{GM(4r - 11GM)}{4r^4}\Theta^2\right] \\
g_{rr} &= \left(1 - \frac{2GM}{r}\right)^{-1} - \frac{GM(2r - 3GM)}{4r^2(r - 2GM)^2}\Theta^2 \\
g_{\theta\theta} &= r^2 + \frac{r^2 - 17GMr + 34G^2M^2}{16r(r - 2GM)}\Theta^2 \\
g_{\phi\phi} &= r^2\sin^2\theta + \frac{(r^2 + 2GMr - 4G^2M^2)\cos^2\theta - 4GM(r - GM)}{16r(r - 2GM)}\Theta^2,
\end{aligned}$$

where the noncommutativity parameter Θ is defined by the commutation relations

$$[\hat{r}, \hat{\theta}] = i\Theta, \quad \texttt{others} = 0. \tag{4}$$

By the way, the commutation relation (4) is different from the usual cartesian one (1) since it corresponds to $\Theta^{ij} = r\Theta\epsilon^{ij}$. However, the Moyal star($\star$)-product (2) can be consistently defined in the spherical coordinates with a constant $\Theta^{r\theta}(\equiv \Theta)$. Therefore, the noncommutativity parameter Θ carries a length dimension. Actually, the metric solution (3) was derived based on the assumption of the spherical representation rather than the cartesian coordinates [9].

Here $G = l_p^2$ and M are the Newton's constant and the total mass of the black hole, respectively. Note that the metric is not spherically symmetric unlike the case of the commutative Schwarzschild black hole in general relativity. It has coordinate singularities, such as apparent and Killing horizons, $\hat{r}_H = 2GM$ and $\tilde{r}_H = 2GM\left(1 + \frac{3\Theta^2}{64G^2M^2}\right)$, respectively. Here, the apparent and Killing horizons are defined by $g^{rr} = g_{rr}^{-1} = 0$ and $g_{tt} = 0$ at each horizon in the usual context, respectively.

Note that there have been some efforts to determine the bounds of the noncommutative parameter [13, 14, 15, 16]. In Refs. [15, 16], it has been shown that the bound of the noncommutativity parameter appears at the order of $10^{-1}l_p$ by taking into account the significant back reaction of the thermal temperature should be the same order of magnitude as the total mass in the Gaussian extension of the point source induced by the noncommutativity. In this paper, we would like to study the bound of the noncommutativity parameter in such a different way that the entropy of the noncommutative Schwarzschild black hole should be satisfied with the well-known area law. The entropy-area relationship in the brick-wall method [17, 18, 19, 20, 21, 22, 23, 24] yields a relation between the brick wall cutoff and the noncommutativity parameter. Remarkably, the real condition of the relation will give the bound of the noncommutativity parameter, which is the same order of that in Refs. [15, 16].

Now, let us consider a scalar field confined in a box near the Killing horizon of the black hole. Along with the metric (3), the equation of motion for the scalar field on this black hole background is,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) - \mu^2\Phi = 0 \tag{5}$$

with the boundary condition, $\Phi(x) = 0 \;\; \texttt{for} \;\; r \leq \tilde{r}_H + \epsilon \; \texttt{and} \;\; r \geq L$ where $\tilde{r}_H, \tilde{r}_H + \epsilon$ and L are the positions of the Killing horizon, the inner and outer walls of a spherical box, respectively. We assumed that a quantum gas is in a thermal equilibrium state at the temperature $T = \beta^{-1}$.

In the WKB approximation with $\Phi = e^{-iEt+iS(r,\theta,\phi)}$, the field equation (5) yields the constraint

$$\frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} + \frac{p_\phi^2}{g_{\phi\phi}} = ((-g^{tt})E^2 - \mu^2), \tag{6}$$

where the momenta p_i's are defined by $p_r = \partial S/\partial r, p_\theta = \partial S/\partial \theta, p_\phi = \partial S/\partial \phi$. Then, according to the semiclassical quantization rule, the number of quantum states $n(E)$ with energy not exceeding E can be written as

$$\begin{aligned} n(E) &= \frac{1}{(2\pi)^3} \int dr d\theta d\phi dp_r dp_\theta dp_\phi \\ &= \frac{1}{6\pi^2} \int dr d\theta d\phi \sqrt{g_{rr} g_{\theta\theta} g_{\phi\phi}} [(-g^{tt} E^2 - \mu^2)]^{3/2}. \end{aligned} \tag{7}$$

Then, the free energy is found to be

$$\begin{aligned} F &= -\int dE \frac{n(E)}{e^{\beta E} - 1} \\ &= -\frac{\pi^2}{90\beta^4} \int_{\tilde{r}_H+\epsilon}^{L} dr (-g^{tt})^{3/2} g_{rr}^{1/2} \int d\theta d\phi (g_{\theta\theta} g_{\phi\phi})^{1/2}, \end{aligned} \tag{8}$$

where the mass of the scalar field is set to be zero for simplicity. In the near horizon, it reads

$$\begin{aligned} F &\simeq -\frac{\pi^2}{90\beta^4} \left[\int_{\tilde{r}_H+\epsilon}^{L} dr (-g^{tt})^{3/2} g_{rr}^{1/2} \right] \left[\int d\theta d\phi (g_{\theta\theta} g_{\phi\phi})^{1/2} \right]_{\tilde{r}_H} \\ &= -\frac{\pi^2 A_H}{90\beta^4} \int_{\tilde{r}_H+\epsilon}^{L} dr (-g^{tt})^{3/2} g_{rr}^{1/2}, \end{aligned} \tag{9}$$

where A_H is the surface area of the noncommutative Schwarzschild black hole on the Killing horizon $\tilde{r}_H$. The degrees of freedom of the scalar field are assumed to be dominated in the vicinity of the horizon, so that one can use the near horizon approximation. To calculate the radial integration of the free energy, it gives

$$F \approx -\frac{8\pi^2 G^2 M^2 (2GM)^{1/2} A_H}{45\beta^4 \Theta^2} \sqrt{\frac{8GM\epsilon + \Theta^2}{\epsilon}}, \tag{10}$$

where we ignored the infra-red contribution coming from the outer boundary L in the radial integration [17]. Now, we need Hawking temperature $\tilde{T}_H$, which is defined by the Killing vector $K = \partial_t$ as

$$\tilde{T}_H = \frac{\tilde{\kappa}_H}{2\pi} = \frac{1}{8\pi GM}\left(1 + \frac{7\Theta^2}{32G^2M^2}\right) = \tilde{\beta}_H^{-1}, \tag{11}$$

where $\tilde{\kappa}_H$ is the surface gravity at the Killing horizon. From now on, we will assume $\Theta^2/G^2M^2 \ll 1$ for later convenience, which means that the black hole mass is much larger than Θ/l_p^2. Using the thermodynamic relation, $S = \beta^2 \partial F/\partial\beta|_{\beta=\tilde{\beta}_H}$, the statistical entropy from (10) simply becomes

$$S_{NS} = \frac{A_H}{180\pi\Theta^2}\sqrt{\frac{\bar{\epsilon}^2 + 2\bar{\epsilon}\Theta + \Theta^2}{\bar{\epsilon}^2 + 2\bar{\epsilon}\Theta}}, \tag{12}$$

where the cutoff parameter ϵ was replaced with a proper length, $\bar{\epsilon} \approx \sqrt{8GM\epsilon + \Theta^2} - \Theta$. The entropy is different from that of the Schwarzschild black hole in that it depends on the noncommutativity parameter.

It is interesting to note that the classical metric solution (3) has a well-defined commutative limit for the vanishing noncommutativity parameter, while the resulting entropy does not. The reason comes from the radial integration of the free energy (9). For instance, just like the integral of a function $e^{\alpha r}$ with a parameter α, the vanishing limit of $\alpha = 0$ is well-defined, however, this is not the case after integration. Similar calculations appear in performing the radial integration in Eq. (9), so that the resulting entropy does not have a smooth vanishing limit.

Let us assume that the entropy (12) is satisfied with the well-known area law even in this noncommutative black hole,

$$S_{NS} = A_H/4l_p^2, \tag{13}$$

which gives the following relation,

$$\bar{\epsilon} = \Theta\left(\sqrt{\frac{\gamma}{\gamma - 1}} - 1\right), \tag{14}$$

where $\gamma = (\sqrt{45\pi}\Theta/l_p)^4$. From the additional condition of $\gamma > 1$ to make the noncommutativity parameter to be real, it is eventually bounded as

$$\Theta > \frac{1}{\sqrt{45\pi}} l_p \sim 8.4 \times 10^{-2} l_p. \tag{15}$$

In particular, there is a critical case of $\gamma = 4/3$ where the noncommutative parameter is the same with the brick wall cutoff.

Now, let us mention the thermodynamic stability of the noncommutative Schwarzschild black hole. Using the thermodynamic relations, one calculate the internal energy of the black hole system as $U = \frac{\partial}{\partial\beta}(\beta F)\Big|_{\tilde{\beta}_H} = \frac{3A_H}{128\pi G^2 M}$. Then, the heat capacity can be evaluated as $C_v = \frac{\partial U}{\partial \tilde{T}_H} = -\frac{3\pi G M^2}{2} I(\theta)\left(1 - \frac{3\Theta^2}{4G^2M^2}\right)$, where the function $I(\theta)$ is explicitly given by $I(\theta) = \int_0^\pi d\theta \frac{\sin^2\theta}{\sqrt{\sin^2\theta + \frac{\Theta^2}{128G^2M^2}(1-19\sin^2\theta)}}$, which is well-defined positive definite function for $\Theta^2/G^2M^2 \ll 1$. As a result, it is negative and thus our noncommutative black hole is thermodynamically unstable as like the commutative case [17].

We have shown that the noncommutative parameter Θ can be bounded by requiring the standard area law of the entropy in the noncommutative Schwarzschild black hole using the brick-wall method. Furthermore, the heat capacity shows that the black hole is still unstable even in spite of the noncommutativity. It is interesting to note that the order of the bound (15) is coincident with $\sqrt{\theta} \gtrsim 10^{-1} l_p$ in Refs. [15, 16], where θ has the squared length dimension.

Acknowledgments

W. KIM was supported by the Special Research Grant of Sogang University, 200911044, and D. LEE was supported by the National Research Foundation of Korea Grant funded by the Korean Government, NRF-2009-351-C00111.

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